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A nonconvex dissipative system and its applications (II)

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Abstract This paper presents exact solutions in terms of implicit functions and hyperbolic functions to a nonconvex dissipative system, controlled by a Duffing–van der Pol nonlinear equation with a fifth-order nonlinearity. Applications to the complex Ginzburg–Landau equation are illustrated and several classes of uniformly translating solutions are obtained accordingly.

Keywords Ginzburg–Landau equation · First integral · Bell-profile wave · Kink-profile wave · Homoclinic trajectory · Heteroclinic trajectory · Ansatz · Uniformly translating solutions

AMS (MOS) subject classification 34C05 · 34C14 · 34C20 · 35B40

1 Introduction

Many physical, biological, and chemical phenomena can be described by nonconvex systems with dissipations. One of the basic problems in the study of nonlinear systems is to find exact solutions and to explicitly describe traveling wave behaviors. Modern theories describe traveling waves and coherent structures in a diverse variety of fields, including general relativity, high-energy particle physics, plasmas, atmosphere and oceans, animal dispersal, random media, chemical reactions, biology, nonlinear

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electrical circuits, and nonlinear optics. For example, in the latter, the mathematics developed for describing the propagation of information via optical solitons is most striking, attaining an incredible accuracy. It has been experimentally verified and spans 12 orders of magnitude: from the wavelength of light to transoceanic distances. It also guides the practical applications in modern telecommunications. Many other nonlinear wave theories mentioned above can claim similar success [1].

By the fact that many nonlinear systems can be converted into nonlinear ordinary differential equations (ODEs) after making proper traveling wave transformations, exploring traveling waves for those nonlinear systems is somehow equivalent to finding exact solutions of the corresponding ODEs. A typical example is the complex Ginzburg–Landau equation (CGLE) [2,3]:

$$\begin{aligned} u_t &= \alpha u + (b_1 + ic_1)u_{xx} - (b_2 - ic_2)|u|^2 u - (b_3 - ic_3)|u|^4 u \\ &\quad + (b_4 + ic_4)(|u|^2 u)_x + (b_5 + ic_5)(|u|^2)_x u, \end{aligned} \tag{1}$$

where $u(x,\ t)$ is a complex function, and b_i, c_i $(i = 1,\ 2, \ldots, 5)$ and α are real coefficients. Assume that Eq. 1 admits a uniformly translating solution in the variable $\xi = x - vt$ of the form

$$\begin{aligned} u(x,\ t) &= \mathrm{e}^{i(kx - wt)} \cdot \hat{u}(x - vt), \\ \hat{u}(\xi) &= a(\xi) \cdot \mathrm{e}^{i\phi(\xi)}, \end{aligned} \tag{2}$$

where ϕ is a real function of the pseudo-time ξ. After substituting (2) into (1) and setting the real part and imaginary part to zero, respectively, we have

$$\begin{aligned} &-b_1 a_{\xi\xi} + (2c_1 D + 2kc_1 - v)a_\xi + b_1 a(\phi_\xi)^2 + 2kb_1 a\phi_\xi + c_1 a\phi_{\xi\xi} \\ &\quad + 2c_1 a_\xi \phi_\xi + c_4 a^3 \phi_\xi + (b_1 k^2 - \alpha)a + (b_2 + c_4 k)a^3 + b_3 a^5 \\ &\quad - (3b_4 + 2b_5)a^2 a_\xi = 0 \end{aligned} \tag{3}$$

and

$$\begin{aligned} &c_1 a_{\xi\xi} + (2kb_1 + 2b_1 D)a_\xi - c_1 a(\phi_\xi)^2 + (v - 2kc_1 a)\phi_\xi + b_1 a\phi_{\xi\xi} \\ &\quad + 2b_1 a_\xi \phi_\xi + b_4 a^3 \phi_\xi + (w - c_1 k^2)a + (c_2 + b_4 k)a^3 + c_3 a^5 \\ &\quad - (3c_4 + 2c_5)a^2 a_\xi = 0. \end{aligned} \tag{4}$$

In order to further simplify Eqs. 3 and 4, we let $\phi(\xi)$ satisfy a Riccati equation

$$\phi'(\xi) = D + Ba^2(\xi), \tag{5}$$

where both D and B are constants to be determined, then Eqs. 3 and 4 reduce to

$$\begin{aligned} &b_1 a_{\xi\xi} + (v - 2kc_1 - 2c_1 D)a_\xi + (3b_4 + 2b_5 - 4Bc_1)a^2 a_\xi \\ &\quad + (\alpha - b_1 D^2 - 2kb_1 D - b_1 k^2)a - (2b_1 DB + 2kb_1 B + c_4 D \\ &\quad + c_4 k + b_2)a^3 - (b_1 B^2 + c_4 + b_3)a^5 = 0 \end{aligned} \tag{6}$$

and

$$\begin{aligned} &c_1 a_{\xi\xi} + (2kb_1 + 2b_1 D)a_\xi + (4Bb_1 + 3c_4 + 2c_5)a^2 a_\xi \\ &\quad + (w - c_1 D^2 + vD - 2kDc_1 - c_1 k^2)a + (vB - 2c_1 DB - 2kc_1 B \\ &\quad + b_4 D + c_2 + b_4 k)a^3 + (b_4 - c_1 B^2 + c_3)a^5 = 0, \end{aligned} \tag{7}$$

respectively.

Note that Eqs. 6 and 7 have exactly the same form as the Duffing–van der Pol oscillator with two polynomial nonlinearities, which was studied in a previous paper [4]

$$\ddot{x} + (\alpha + \beta x^2)\dot{x} - \gamma x + x^3 - \mu x^5 = 0, \tag{8}$$

where α, β, γ, and μ are real constants, and an overdot denotes differentiation with respect to time. When $\alpha\beta < 0$, the existence of the limit cycle has been described in [5]. Numerical simulations indicate that this system is very sensitive to the given initial conditions and the dual solution plays an important role in understanding the behavior of this nonconvex dynamical system [6,7]. Due to the appearance of a fifth-order nonlinearity, solving Eq. 8 for exact solutions becomes more complicated and it does not seem that the detailed study on this problem has been presented previously. Therefore, in this work we first restrict our attention to exact solutions of Eq. 8, then we apply both qualitative analysis presented in the preceding work [4] and analytical results described herein to the study of the uniformly translating solution of CGLE (1).

The rest of the paper is organized as follows. In Sect. 2, exact solutions of Eq. 8 in terms of implicit functions and hyperbolic functions are established by analyzing homoclinic or heteroclinic trajectories of the corresponding equivalent systems, which agree with qualitative analysis illustrated in [4]. In Sect. 3, more explicit exact solutions in terms of hyperbolic functions and trigonometrical functions are found under various parametric conditions by using a general ansatz. Applications of these solutions to CGLE are demonstrated in Sect. 4 and several classes of uniformly translating solutions are derived accordingly. In Sect. 5, we give a brief discussion.

2 Exact solutions in terms of implicit functions and hyperbolic functions

In many physical phenomena, only bounded analytical solutions of modeled systems possess physical meaning and applications. Hence, in this section, it is reasonable for us to start our study by limiting the attention to bounded exact solutions of Eq. 8, even though unbounded exact solutions can be derived in the same manner.

Using the first equation in system 6 [4, p. 6], Eq. 8 can actually be reduced to an Abel equation of the second kind

$$yy' + (\alpha + \beta x^2)y - \gamma x + x^3 - \mu x^5 = 0, \tag{9}$$

where $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Continuing changing dependent and independent variables in (9) as

$$y\,\mathrm{d}t = \mathrm{d}x, \quad x = -\rho \cdot \exp[(-1/\beta)t], \quad t = -\frac{\beta}{2}\ln\eta \tag{10}$$

when $\alpha \cdot \beta = 4$ and $\beta^2 \cdot \gamma = -3$ hold, Eq. 9 becomes

$$\rho''(\eta) - \frac{\beta^2}{2}\rho^2\rho'(\eta) - \frac{\beta^2\mu}{4}\rho^5 = 0. \tag{11}$$

From Appendix A in [4], when $\mu = 0$, Eq. 11 has an implicit solution as

$$\eta = \frac{\sqrt[3]{6c/\beta^2}}{3c}\left[\frac{1}{2}\ln\left(\frac{\rho^2 + 2d\rho + d^2}{\rho^2 - d\rho + \rho^2}\right) - \sqrt{3}\arctan\left(\frac{\sqrt{3}\rho}{\rho - 2d}\right)\right] + \eta_0, \tag{12}$$

where c and η_0 are integration constants, and $d = \sqrt[3]{6c/\beta^2}$. The second term in (12) can be expressed as a logarithmic function, then we have

$$\left[\frac{\rho+d}{\rho^2-d\rho+d^2}\right]\left[\frac{(1+\sqrt{3}i)\rho-2d}{(1-\sqrt{3}i)\rho-2d}\right]=T\exp\left[\frac{6c}{d}(\eta-\eta_0)\right], \tag{13}$$

where $T = 1/d$.

Making use of transformation (10) and changing to the original variables, we obtain that when $\alpha\cdot\beta = 4$ and $\beta^2\cdot\gamma = -3$ hold, Eq. 8 has an exact solution as

$$x = -\rho\cdot\exp[(-1/\beta)t], \tag{14}$$

where ρ is determined by the implicit function (13) and $\eta = \exp(-2/\beta)t$.

From Appendix A again, when $\mu = 0$, Eq. 11 has one first integral of the form $\rho'(\eta) = \frac{\beta^2}{6}\rho^3(\eta) + c$. Choosing $c = 0$, we get a particular solution to Eq. 8:

$$x = \pm\frac{\sqrt{-3C_1}\mathrm{e}^{(1/\beta)t}}{\beta\sqrt{C_1\mathrm{e}^{(2/\beta)t}+C_2}}, \tag{15}$$

where both C_1 and C_2 are arbitrary constants.

The above results are in agreement with the qualitative analysis undertook in the preceding paper, in which we derived that there are two particular cases to Eq. 8:

Case 1 When $\mu = 0$ and $\beta \neq 0$, under the parametric choices $\alpha\cdot\beta = 4$ and $\beta^2\cdot\gamma = -3$, Eq. 8 has one first integral

$$\mathrm{e}^{6t/\beta}\left((\dot{x})^2+\dot{x}\left[\frac{2}{3}\beta x^3+\frac{2x}{\beta}+c_1\mathrm{e}^{-3t/\beta}\right]\right.$$
$$\left.+x^2\left[\frac{1}{9}\beta^2x^4+\frac{2}{3}x^2+\frac{1}{\beta^2}\right]+c_1x\mathrm{e}^{-3t/\beta}\left[\frac{1}{3}\beta x^2+\frac{1}{\beta}\right]\right)=c_2,$$

where c_1 and c_2 are arbitrary. Taking $c_1 = 2c$ and $c_2 = -c^2$, we have a particular case:

$$\mathrm{e}^{(3/\beta)t}\left(\dot{x}+\frac{\beta}{3}x^3+\frac{x}{\beta}\right)+c=0.$$

Thus, Eq. 8 reduces to an Able's equation of the fist kind

$$\dot{x}+\frac{1}{3}\beta x^3+\frac{1}{\beta}x=-c\mathrm{e}^{-\frac{3}{\beta}t}. \tag{16}$$

When $c = 0$, solving Eq. 16 gives an exact solution as (15); when $c \neq 0$, solving Eq. 16 gives another exact solution as (14) which contains the implicit function (13).

Case 2 When $\alpha = 0$ and $\beta = 0$, system (6) [4, p. 6] is degenerated to

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=\gamma x-x^3+\mu x^5.\end{aligned} \tag{17}$$

Note that system (17) is symmetric about $O(0,\ 0)$, the x-axis, and the y-axis, respectively. Each bounded exact solution to Eq. 8 in this case corresponds to a homoclinic or heteroclinic trajectory of system (17). Thus, in order to find the explicit exact solution to Eq. 8, according to the qualitative theory and bifurcation theory of dynamical systems [8–10], we need to analyze homoclinic or heteroclinic trajectories of system (17). We know that system (17) has one first integral as

$$\dot{x}^2 = \gamma x^2 - \frac{x^4}{2} + \frac{\mu x^6}{3} + K,$$

where K is arbitrary real constant. Next, for the convenience of our statement, we denote

$$p_\pm = \begin{cases} \pm\sqrt{\frac{\sqrt{1-4\gamma\mu}-1}{-2\mu}}, & \mu \neq 0, \\ \pm\sqrt{\gamma}, \quad \mu = 0, & \gamma > 0 \end{cases}$$

and

$$q_\pm = \pm\sqrt{\frac{\sqrt{1-4\gamma\mu}+1}{2\mu}}, \quad \mu > 0.$$

Due to the fact that system (17) is symmetrical, we only need to take discussions on the right half plane $\{(x, y)|x \geq 0, y \in \mathbb{R}\}$. Applying the qualitative theory of planar dynamical systems, we find that (1) when $\mu = 0, \gamma > 0$ or $\mu < 0, \gamma > 0$, there are two equilibrium points on the right x-axis, namely, a saddle point $O(0, 0)$ and a center point $A_+(p_+, 0)$; (2) when $\mu > 0$ and $\gamma < 0$, $O(0, 0)$ is a saddle point and $B_+(q_+, 0)$ is a saddle point; and (3) when $\mu > 0, \gamma > 0$ and $1 - 4\mu\gamma > 0$, $O(0, 0)$ is a saddle point, $A_+(p_+, 0)$ is a center point and $B_+(q_+, 0)$ is another saddle point. If $1 = 4\mu\gamma$, $O(0, 0)$ is still a saddle point $O(0, 0)$, but A_+ and B_+ get coincided and change to a degenerated equilibrium point. Specifically, we have

(1) In the case of $\mu = 0, \gamma > 0$ or $\mu < 0, \gamma > 0$, system (17) has two homoclinic trajectories emanating from the saddle O: $y^2 = \gamma x^2 - \frac{x^4}{2} + \frac{\mu x^6}{3}$. Combining this formula with system (17), we obtain an exact solution

$$x(t) = \pm \left\{ \frac{\gamma\sqrt{\frac{3}{3-16\mu\gamma}}\operatorname{sech}^2\sqrt{\gamma}(t-t_0)}{\frac{1}{2} + \left(\frac{1}{4}\sqrt{\frac{3}{3-16\mu\gamma}} - \frac{1}{4}\right)\operatorname{sech}^2\sqrt{\gamma}(t-t_0)} \right\}^{1/2}, \tag{18}$$

where t_0 is arbitrary.

(2) In the case of $\mu > 0$ and $\gamma < 0$, system (17) has two heteroclinic trajectories connecting $B_+(q_+, 0)$ and $B_-(q_-, 0)$: $y^2 = \gamma x^2 - \frac{x^4}{2} + \frac{\mu x^6}{3} + K_1$, where $K_1 = -\gamma q_+^2 + \frac{q_+^4}{2} - \frac{\mu q_+^6}{3}$ and $q_- \leq x \leq q_+$. Combining this formula with system (17), we obtain an exact solution

$$x(t) = q_+ \left\{ \frac{1 - \operatorname{sech} S(t-t_0)}{1 - \nu \operatorname{sech} S(t-t_0)} \right\}^{1/2}, \tag{19}$$

where

$$S = \sqrt{2}q_+\sqrt[4]{1-4\mu\gamma}, \qquad \nu = \frac{1+4\sqrt{1-4\mu\gamma}}{-1+2\sqrt{1-4\mu\gamma}}. \tag{20}$$

(3) In the case of $\mu > 0, \gamma > 0$ and $1-4\mu\gamma > 0$, there are three subcases for system (17): (a) when $16\mu\gamma - 3 = 0$, there are two heteroclinic trajectories connecting $O(0,\ 0)$ and $B_+(q_+,\ 0)$ or $B_-(q_-,\ 0)$, and the associated exact solutions are

$$x(t) = \pm\left\{2\gamma\left[1 \pm \tanh\sqrt{\gamma}(t-t_0)\right]\right\}^{1/2}; \tag{21}$$

(b) when $16\mu\gamma - 3 < 0$, there are two homoclinic trajectories terminating at $O(0,\ 0)$ and two heteroclinic trajectories connecting $B_+(q_+,\ 0)$ and $B_-(q_-,\ 0)$. The forms of associated exact solutions are the same as (18) and (19), respectively; (c) when $16\mu\gamma - 3 > 0$, there are two homoclinic trajectories connecting $B_+(q_+,\ 0)$ and $B_-(q_-,\ 0)$, and the associated exact solutions are

$$x(t) = q_+\left\{\frac{1 + \operatorname{sech} S(t-t_0)}{1 + \nu \operatorname{sech} S(t-t_0)}\right\}^{1/2}, \tag{22}$$

where S and ν are the same as given in (20).

3 A general ansatz

In this section, we continue our study of constructing exact solutions to Eq. 8 and express them explicitly in terms of hyperbolic functions and trigonometrical functions. Before our discussion, let us give a preliminary introduction of two special wave solutions. Suppose that Eq. 8 has a solution $x(t)$ which satisfies

$$x'(t) \to 0, \qquad x''(t) \to 0 \ \text{ as } \ |t| \to +\infty \tag{23}$$

and has the horizontal asymptotes as

$$\lim_{t\to\pm\infty} x(t) = D_\pm, \tag{24}$$

which are two roots of the algebraic equation

$$\mu x^5(t) - x^3(t) + \gamma x(t) = 0. \tag{25}$$

Note that the roots of Eq. 25 are actually the x-coordinates of regular equilibrium points of system (6) [4, p6] on the x-axis in the Poincaré phase plane. Following the traditional definition: when $D_+ = D_-$, we call $x(t)$ the bell-profile wave solution, and when $D_+ \neq D_-$, we call $x(t)$ the kink-profile wave solution. Both types of wave solutions play a crucial role and have wide applications in modern theoretical physics. In the early 1960s, one particularly noteworthy contribution was the explosion of activity unleashed by the numerical discovery of the soliton (solitary wave) by Zabusky and Kruskal [11], Zabusky and Galvin [12], and Kruskal [13] a name intended to signify particle-like quantities, and the earliest theoretical explanation by Gardner, Greene, Kruskal, and Miura in 1960s and early 1970s [14–16], which subsequently led to the present-day theory of integrable partial differential equations. Nonlinear waves and coherent structures is an inter-disciplinary area that has many significant applications, including nonlinear optics, hydrodynamics, plasmas and solid-state physics. In fact, for any physical system where the dynamics is driven by, and mainly determined by, phase coherence of the individual waves, it has many useful applications. For instance, the Klein–Gordon equation has applications in quantum field theory and the KdV-type equations are modeled from plasma physics and solid-state physics. Both of them have kink-profile wave solutions which have been used to calculate energy and momentum flow and topological charge in the quantum field [17,18].

Multiplying both sides of Eq. 8 by $x'(t)$ and integrating it from $-\infty$ to t gives

$$\frac{1}{2}[x'(t)]^2+\int_{-\infty}^{t}[\alpha+\beta x^2(\xi)][x'(\xi)]^2\mathrm{d}\xi-\frac{\gamma}{2}x^2(t)+\frac{1}{4}x^4(t)-\frac{\mu}{6}x^6(t)=c_1,$$

where c_1 is integration constant. Letting $t\to+\infty$ and $-\infty$, respectively, we have

$$\int_{-\infty}^{+\infty}[\alpha+\beta x^2(\xi)][x'(\xi)]^2\mathrm{d}\xi-\frac{\gamma}{2}D_+^2+\frac{1}{4}D_+^4-\frac{\mu}{6}D_+^6=c_1 \tag{26}$$

and

$$-\frac{\gamma}{2}D_-^2+\frac{1}{4}D_-^4-\frac{\mu}{6}D_-^6=c_1. \tag{27}$$

From (26) and (27), we obtain

$$\int_{-\infty}^{+\infty}[\alpha+\beta x^2(\xi)][x'(\xi)]^2\mathrm{d}\xi=\frac{\gamma}{2}(D_+^2-D_-^2)-\frac{1}{4}(D_+^4-D_+^4)+\frac{\mu}{6}(D_+^6-D_+^6). \tag{28}$$

Using (25), we have

$$\mu D_+^5-D_+^3+\gamma D_+=0 \quad\text{and}\quad \mu D_-^5-D_-^3+\gamma D_-=0,$$

which gives

$$D_+^4-D_-^4=\mu(D_+^6-D_-^6)+\gamma(D_+^2-D_-^2).$$

Hence, Eq. 28 can be reduced as

$$\int_{-\infty}^{+\infty}[\alpha+\beta x^2(\xi)][x'(\xi)]^2\mathrm{d}\xi=\frac{\gamma}{4}(D_+^2-D_-^2)-\frac{\mu}{12}(D_+^6-D_+^6). \tag{29}$$

Equation 29 indicates that when $\alpha\beta>0$, if $x(t)$ is an bounded exact solution to Eq. 8 satisfying assumptions (23)–(25), then

$$\alpha \quad\text{and}\quad \left[\frac{\gamma}{4}(D_+^2-D_-^2)-\frac{\mu}{12}(D_+^6-D_+^6)\right]$$

must have the same sign. Furthermore, from (29) again, one can see that when α is not equal to zero, Eq. 8 does not have bell-profile wave solutions, but kink-profile wave solutions. This conclusion is helpful when we seek exact solutions of Eq. 8.

Now we use a general ansatz to construct implicit exact solutions to Eq. 8. This ansatz was originally introduced by Fan et al. [19] and Yan [20] when they used it to seek traveling solitary wave solutions to the Burgers-KdV equation and integrable coupled nonlinear evolution equations, and modified by some researchers, for instance, Li et al. [21] and Lu and Fan [22] when they investigated several KdV-type equations. The basic idea is that for a given nonlinear ordinary differential equation

$$G\big(u,\,u',\,u'',\,u''',\ldots,u^{(l)}\big)=0, \tag{30}$$

where a prime denotes $\frac{\mathrm{d}}{\mathrm{d}\xi}$. In order to find explicit exact solutions of Eq. 30, we take the following ansatz:

$$u(\xi)=\sum_{i=1}^{m}w^{i-1}(\xi)\left[A_iw(\xi)+B_i\sqrt{R+w^2(\xi)}\right]+A_0. \tag{31}$$

Here we assume that the new variable $w = w(\xi)$ satisfies a Riccati equation

$$\frac{dw}{d\xi} = R + w^2, \tag{32}$$

where A_0, A_i, B_i $(i = 1, 2, \ldots, m)$ and R are constants to be determined, and m is a positive integer. However, after substitution of (31) into Eq. 30 and setting the coefficient of the highest-order partial derivative term to the one of the highest-order nonlinearity term, usually one can find that the constant m is not a natural number. That is, m can be equal to either a fraction or a negative integer. In this case, we need to make the following transformation prior to performing substitution.

(1) When $m = q/p$ (p and q are relatively prime), we let

$$u(\xi) = \chi^{q/p}(\xi), \tag{33}$$

then substitute (33) into Eq. 30 and solve the resulting equation for the new value of m of Eq. 31 by balancing the highest-order partial derivative term and the leading nonlinear term.

(2) When m is a negative integer, we let

$$u(\xi) = \chi^{m}(\xi), \tag{34}$$

then substitute (34) into Eq. 30 and similarly solve the resulting equation for the new value of m of Eq. 31 again. In general, the constant m can be converted into a positive integer by means of the above transformation. Otherwise we have to seek other proper transformations.

The procedure of the general ansatz can be summarized as follows:

- **Step 1** Determine the values of m in (31) by balancing the highest-order partial derivative term and the leading nonlinear term in (30). Specifically, (1) If m is a positive integer then go to Step 2; (2) If $m = q/p$, p and q are integers and relatively prime, we make the transformation (33) and then go back to Step 1; (3) If m is a negative integer, we make the transformation (34) and then go back to Step 1.
- **Step 2** With the aid of mathematical softwares such as Maple, Matlab, and Mathematica, substituting (31) along with the condition (32) into Eq. 30 yields a system of algebraic equations with respect to $w^i(R+w^2)^{j/2}$ $(j = 0, 1, \ldots; i = 0, 1, 2, \ldots)$.
- **Step 3** Collect all terms with the same power in $w^i(R+w^2)^{j/2}$ $(j = 0, 1, \ldots; i = 0, 1, 2, \ldots)$. Set the coefficients of the terms $w^i(R+w^2)^{j/2}$ $(j = 0, 1, \ldots; i = 0, 1, 2, \ldots)$ to zero, and then obtain an over-determined system of nonlinear algebraic equations with respect to the unknown variables R, A_0, A_i, B_i $(i = 1, 2, \ldots, m)$.
- **Step 4** Using the mathematical software again and applying the Wu-elimination method [23] to solve the above over-determined system of nonlinear algebraic equations generated in Step 3, we obtain the values of R, A_0, A_i, B_i $(i = 1, 2, \ldots, m)$.
- **Step 5** It is well known that the general solution of Eq. 32 can be classified as

  (1) when $R < 0$,

$$w(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi), \qquad w(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\xi). \tag{35}$$

(2) when $R = 0$,

$$w(\xi) = -\frac{1}{\xi}.$$

(3) when $R > 0$,

$$w(\xi) = \sqrt{-R}\tan(\sqrt{R}\xi), \qquad w(\xi) = -\sqrt{R}\cot(\sqrt{R}\xi). \tag{36}$$

Therefore, according to Steps 1–5 and making use of the values of R, A_0, A_i, B_i ($i = 1, 2, \ldots, m$) obtained in Step 4, we can construct a class of exact solutions of Eq. 30 with various conditions.

Reverting to Eq. 8 and following Step 1, after balancing $\ddot{x}$ and x^5, we derive $m = 1/2$. Thus, making the transformation

$$x(t) = h^{1/2}(t) \tag{37}$$

and substituting (37) into Eq. 8, we have

$$2h(t)h''(t) + 2\beta h^2(t)h'(t) - [h'(t)]^2 + 2\alpha h(t)h'(t) - 4\gamma[h(t)]^2 + 4[h(t)]^3 - 4\mu[h(t)]^4 = 0. \tag{38}$$

Suppose that Eq. 38 admits the solution of the form (31). When $\beta = 0$, after substituting (31) into (38) and equating the coefficients of $h(t)h''(t)$ and $h^4(t)$, we derive $m = 1$ in formula (31). That is, we assume that the solution of Eq. 38 has the form

$$h(t) = B_1\sqrt{R + w^2} + A_1 w + A_0, \tag{39}$$

where $w(t)$ is a solution of Eq. 32, and R, B_1 and A_i ($i = 0, 1$) are to be determined. Substituting (39) into (38) and equating the coefficients of $w^i(R + w^2)^{l/2}$ ($i = 1, 2, 3, 4$, $l = 0, 1$), we get a resulting algebraic system with unknowns B_1, A_i ($i = 0, 1$), and R.

$$\begin{aligned}
&2A_0A_1 + \alpha(A_1^2 + B_1^2) + 6A_1B_1^2 - 24\mu A_1A_0B_1^2 = 0,\\
&\alpha A_0A_1 - 2\gamma B_1^2 + 6B_1^2A_0 - 12\mu A_0^2B_1^2 = 0,\\
&-2\gamma A_0B_1 + 3A_0^2B_1 - 4\mu A_0^3B_1 = 0,\\
&\alpha A_0 - 4\gamma A_1 + 12A_1A_0 - 24\mu A_1A_0^2 = 0,\\
&6A_1^2B_1 + A_0B_1 - 24\mu A_1^2A_0B_1 + \alpha A_1B_1 = 0,\\
&A_0B_1 + 2B_1^3 + \alpha A_1B_1 - 8\mu A_0B_1^3 = 0.
\end{aligned}$$

Solving the above system consistently with aid of maple when $16\mu\gamma - 3 + \alpha^2 = 0$, we have two cases as follows

Case 1 $A_1 = \pm\frac{1}{4}\sqrt{\frac{3}{\mu}}$, $\quad A_0 = \frac{3\mp\alpha\sqrt{3\mu}}{8\mu}$, $\quad B_1 = \pm\frac{1}{4}\sqrt{\frac{3}{\mu}}$, $\quad R = -\frac{16}{3}\mu A_0^2$;

Case 2 $A_1 = \pm\frac{1}{2}\sqrt{\frac{3}{\mu}}$, $\quad A_0 = \frac{3\mp\alpha\sqrt{3\mu}}{8\mu}$, $\quad B_1 = 0$, $\quad R = -\frac{4}{3}\mu A_0^2$.

Hence, according to (35) and (36), from Case 1, when $\mu > 0$ we obtain that Eq. 8 has exact solutions:

$$x_1(t) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu}\left(1 \pm \tanh\left[\sqrt{-R}(t-t_0)\right] \pm i\,\mathrm{sech}\left[\sqrt{-R}(t-t_0)\right]\right)\right]^{1/2}, \tag{40}$$

$$x_2(t) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu}\left(1 \pm \coth\left[\sqrt{-R}(t-t_0)\right] \pm i\,\mathrm{cosech}\left[\sqrt{-R}(t-t_0)\right]\right)\right]^{1/2} \tag{41}$$

and when $\mu < 0$ Eq. 8 has exact solutions:

$$x_3(t) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu}\left(1 \pm i\tan\left[\sqrt{R}(t-t_0)\right] \pm i\sec\left[\sqrt{R}(t-t_0)\right]\right)\right]^{1/2}, \tag{42}$$

$$x_4(t) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu}\left(1 \pm i\cot\left[\sqrt{R}(t-t_0)\right] \pm i\,\mathrm{cosec}\left[\sqrt{R}(t-t_0)\right]\right)\right]^{1/2}, \tag{43}$$

where R is the same as in Case 1, and t_0 is an arbitrary constant.

Similarly, for Case 2, when $\mu > 0$ we obtain that Eq. 8 has exact solutions:

$$x_5(t) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu}\left(1 \pm \tanh\left[\sqrt{-R}(t-t_0)\right]\right)\right]^{1/2}, \tag{44}$$

$$x_6(t) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu}\left(1 \pm \coth\left[\sqrt{-R}(t-t_0)\right]\right)\right]^{1/2}, \tag{45}$$

and when $\mu < 0$ Eq. 8 has exact solutions:

$$x_7(t) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu}\left(1 \pm i\tan\left[\sqrt{R}(t-t_0)\right]\right)\right]^{1/2}, \tag{46}$$

$$x_8(t) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu}\left(1 \pm i\cot\left[\sqrt{R}(t-t_0)\right]\right)\right]^{1/2}, \tag{47}$$

where R is the same as given in Case 2, and t_0 is an arbitrary constant.

It is remarkable that formula (21) in the preceding section is identical to (44) when $\alpha = 0$. Apparently, unbounded solution (45) is also able to be derived through analyzing the corresponding homoclinic or heteroclinic trajectory of system (17).

4 Uniformly translating solutions to CGLE

We note that if the ratio of the corresponding coefficients in Eqs. 6 and 7 is equal to each other, then Eq. 6 becomes identical to Eq. 7, and both of them have the same form as the Duffing–van der Pol equation (8). Hence all arguments and results of Eq. 8 can be applied. In this section, we return to CGBE (1) and are going to construct uniformly translating solutions to CGLE (1) based on explicit and implicit results obtained in the last two sections.

In the following discussion, we assume that all coefficients of Eqs. 1 and 5 satisfy

$$\begin{aligned}\frac{b_1}{c_1} &= \frac{v-2kc_1-2c_1D}{2kb_1+2b_1D} = \frac{3b_4+2b_5-4Bc_1}{4Bb_1+3c_4+2c_5}\\ &= \frac{\alpha-b_1D^2-2kb_1D-b_1k^2}{w-c_1D^2+vD-2kDc_1-c_1k^2} = -\frac{b_1B^2+c_4+b_3}{b_4-c_1B^2+c_3}\\ &= -\frac{2b_1DB+2kb_1B+c_4D+c_4k+b_2}{vB-2c_1DB-2kc_1B+b_4D+c_2+b_4k}\end{aligned} \tag{48}$$

and

$$\frac{2b_1DB+2kb_1B+c_4D+c_4k+b_2}{b_1} = -1. \tag{49}$$

Denote that

$$\begin{aligned}\alpha &= \frac{v-2kc_1-2c_1D}{b_1}, \qquad \beta = \frac{3b_4+2b_5-4Bc_1}{b_1},\\ \gamma &= \frac{b_1D^2+2kb_1D+b_1k^2-\alpha}{b_1}, \qquad \mu = \frac{b_1B^2+c_4+b_3}{b_1}\ .\end{aligned} \tag{50}$$

Following formulas (14), (15), (18)–(22), and (40)–(47), we obtain several classes of uniformly translating solutions to CGBE (1) immediately as follows:

Case 1 When $b_1B^2+c_4+b_3=0$, $b_4-c_1B^2+c_3=0$, $\alpha\beta=4$ and $\beta^2\gamma=-3$, CGLE (1) has uniformly translating solutions of form (2), namely

$$u(x,\ t) = \mathrm{e}^{i(kx-wt)}\cdot\hat{u}(x-vt),$$
$$\hat{u}(\xi) = a(\xi)\cdot\mathrm{e}^{i\phi(\xi)}, \quad \xi = x-vt,$$

where $k,\ w,\ v$ satisfy (48) and (49), $\phi(\xi)$ is determined by Eq. 5, and $a(\xi)$ is given by

$$a_1(\xi) = -\rho(\xi)\exp[(-1/\beta)\xi],$$

where ρ is determined by the implicit function

$$\left[\frac{\rho+d}{\rho^2-d\rho+d^2}\right]\left[\frac{(1+\sqrt{3}i)\rho-2d}{(1-\sqrt{3}i)\rho-2d}\right] = T_0\exp\left[\frac{6c}{d}\left(\exp[(-2/\beta)\xi]-\xi_0\right)\right],$$

where $T_0=\sqrt[3]{\beta^2/(6c)}$ (here c is nonzero constant). Particularly, when c is zero, $a(\xi)$ becomes

$$a_2(\xi) = \pm\frac{\sqrt{-3C_1}\mathrm{e}^{(1/\beta)\xi}}{\beta\sqrt{C_1\mathrm{e}^{(2/\beta)\xi}+C_2}}\quad ,$$

where both C_1 and C_2 are arbitrary constants.

Case 2 When $\alpha=\beta=0$ in (50), and either $b_1B^2+c_4+b_3=0$, $b_4-c_1B^2+c_3=0$ and $b_1(\alpha-b_1D^2-2kb_1D-b_1k^2)<0$, or $b_1(b_1B^2+c_4+b_3)<0$, and $b_1(\alpha-b_1D^2-2kb_1D-b_1k^2)<0$, CGLE (1) has uniformly translating solutions of form (2), where $k,\ w,\ v$ satisfy (48) and (49), $\phi(\xi)$ is determined by Eq. 5, and $a(\xi)$ is given by

$$a_3(\xi) = \pm \left\{ \frac{\gamma\sqrt{\frac{3b_1^2}{3b_1^2-16(b_1B^2+c_4+b_3)(b_1D^2+2kb_1D+b_1k^2-\alpha)}}\operatorname{sech}^2\sqrt{\gamma}(\xi-\xi_0)}{\frac{1}{2}+\left(\frac{1}{4}\sqrt{\frac{3b_1^2}{3b_1^2-16(b_1B^2+c_4+b_3)(b_1D^2+2kb_1D+b_1k^2-\alpha)}}-\frac{1}{4}\right)\operatorname{sech}^2\sqrt{\gamma}(\xi-\xi_0)} \right\}^{1/2},$$

where γ is the same as given in (50) and ξ_0 is an arbitrary constant.

Case 3 When $\alpha = \beta = 0$ in (50), $b_1(b_1B^2 + c_4 + b_3) > 0$ and $b_1(b_1D^2 + 2kb_1D + b_1k^2 - \alpha) < 0$, CGLE (1) has uniformly translating solutions of form (2), where $\phi(\xi)$ is determined by Eq. 5 and $a(\xi)$ is given by

$$a_4(\xi) = \sqrt{\frac{\sqrt{1-4\gamma\mu}+1}{2\mu}} \left\{ \frac{1-\operatorname{sech} S(\xi-\xi_0)}{1-\nu \operatorname{sech} S(\xi-\xi_0)} \right\}^{1/2},$$

where

$$S = \sqrt[4]{4(1-4\mu\gamma)}\sqrt{\frac{\sqrt{1-4\gamma\mu}+1}{2\mu}}, \qquad \nu = \frac{1+4\sqrt{1-4\mu\gamma}}{-1+2\sqrt{1-4\mu\gamma}}, \tag{51}$$

here μ and γ are the same as given in (50).

Case 4 When $\alpha = \beta = 0$ in (50), $b_1(b_1B^2+c_4+b_3) > 0$, $b_1(b_1D^2+2kb_1D+b_1k^2-\alpha)>0$ and μ, γ in (50) satisfy $1/4 > \mu\gamma > 3/16$, CGLE (1) has uniformly translating solutions of form (2), where $\phi(\xi)$ is determined by Eq. 5 and $a(\xi)$ is given by

$$a_5(\xi) = \sqrt{\frac{\sqrt{1-4\gamma\mu}+1}{2\mu}} \left\{ \frac{1+\operatorname{sech} S(\xi-\xi_0)}{1+\nu \operatorname{sech} S(\xi-\xi_0)} \right\}^{1/2},$$

where S and ν are the same as given in (51).

Case 5 When μ, γ in (50) satisfy $\alpha^2 + 16\mu\gamma - 3 = 0$ and $b_1(b_1B^2 + c_4 + b_3) > 0$, CGLE (1) has uniformly translating solutions of form (2), where $\phi(\xi)$ is determined by Eq. 5 and $a(\xi)$ is given by either

$$a_6(\xi) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu} \left(1 \pm \tanh\left[\sqrt{-R}(\xi-\xi_0)\right] \pm i \operatorname{sech}\left[\sqrt{-R}(\xi-\xi_0)\right] \right) \right]^{1/2}$$

or

$$a_7(\xi) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu} \left(1 \pm \coth\left[\sqrt{-R}(\xi-\xi_0)\right] \pm i \operatorname{cosech}\left[\sqrt{-R}(\xi-\xi_0)\right] \right) \right]^{1/2}$$

where $R = -\frac{16}{3}\mu\left(\frac{3\mp\alpha\sqrt{3\mu}}{8\mu}\right)^2$ and ξ_0 is an arbitrary constant.

Case 6 When μ, γ in (50) satisfy $\alpha^2 + 16\mu\gamma - 3 = 0$ and $b_1(b_1B^2 + c_4 + b_3) < 0$, CGLE (1) has uniformly translating solutions of form (2), where $\phi(\xi)$ is determined by Eq. 5 and $a(\xi)$ is given by either

$$a_8(\xi) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu} \left(1 \pm i \tan\left[\sqrt{R}(\xi-\xi_0)\right] \pm i \sec\left[\sqrt{R}(\xi-\xi_0)\right] \right) \right]^{1/2}$$

or

$$a_9(\xi) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu}\left(1 \pm i\cot\left[\sqrt{R}(\xi - \xi_0)\right] \pm i\,\mathrm{cosec}\left[\sqrt{R}(\xi - \xi_0)\right]\right)\right]^{1/2},$$

where $R = -\frac{16}{3}\mu\left(\frac{3\mp\alpha\sqrt{3\mu}}{8\mu}\right)^2$ and ξ_0 is an arbitrary constant.

Case 7 When μ, γ in (50) satisfy $\alpha^2 + 16\mu\gamma - 3 = 0$ and $b_1(b_1B^2 + c_4 + b_3) > 0$, CGLE (1) has uniformly translating solutions of form (2), where $\phi(\xi)$ is determined by Eq. 5 and $a(\xi)$ is given by either

$$a_{10}(\xi) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu}\left(1 \pm \tanh\left[\sqrt{-R}(\xi - \xi_0)\right]\right)\right]^{1/2},$$

or

$$a_{11}(\xi) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu}\left(1 \pm \coth\left[\sqrt{-R}(\xi - \xi_0)\right]\right)\right]^{1/2},$$

where $R = -\frac{4}{3}\mu\left(\frac{3\mp\alpha\sqrt{3\mu}}{8\mu}\right)^2$ and ξ_0 is an arbitrary constant.

Case 8 When μ, γ in (50) satisfy $\alpha^2 + 16\mu\gamma - 3 = 0$ and $b_1(b_1B^2 + c_4 + b_3) < 0$, CGLE (1) has uniformly translating solutions of form (2), where $\phi(\xi)$ is determined by Eq. 5 and $a(\xi)$ is given by either

$$a_{12}(\xi) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu}\left(1 \pm i\tan\left[\sqrt{R}(\xi - \xi_0)\right]\right)\right]^{1/2}$$

or

$$a_{13}(\xi) = \left[\frac{3 \mp \alpha\sqrt{3\mu}}{8\mu}\left(1 \pm i\cot\left[\sqrt{R}(\xi - \xi_0)\right]\right)\right]^{1/2},$$

where $R = -\frac{4}{3}\mu\left(\frac{3\mp\alpha\sqrt{3\mu}}{8\mu}\right)^2$ and ξ_0 is an arbitrary constant.

5 Discussion

A great number of nonconvex dissipative systems are known to display solutions we can call "coherent structures". These states are either themselves localized in space or they consist of a spatially extended regular pattern with a localized defect. Examples are kink-profile waves and bell-profile waves in one-dimensional systems, and targets, spirals, dislocations or grain boundaries in two-dimension cases. Such structures have been identified in experiments on thermal convection in pure fluids and binary mixtures [24,25], on parametric surface waves in fluids [26], on the onset of oscillatory convection in binary fluid mixtures [27,28], in nonlinear light-wave propagation in fibers [29], and on nonlinear traveling wave convection in a narrow annular cell [30]. They play an important role in the dynamics of nonconvex systems, for example in the selection of a final steady pattern at long times, and in the time evolution of periodic, quasiperiodic or disordered (chaotic) patterns.

A simple set of models that account for this type of behavior are the CGLE or its generalizations, of which a prototype is Eq. 1 without the last two terms. The present study provides a connection between Eq. 1 and a Duffing–van der Pol equation (8). It first confines itself to explicit and implicit exact solutions of the Duffing–van der Pol equation with two polynomial nonlinearities by means of analyzing the corresponding homoclinic or heteroclinic trajectories and a general ansatz, and then focuses on the application to CGLE. Several classes of uniformly translating solutions are constructed.

It is worthwhile to mention that the application does not depend on the particular example of CGLE. One can apply the techniques and results described in this and a companion article [4] (referred to as Paper I, in the preceding) to the study of some other nonlinear differential equations, for instance,

(1) Generalized derivative Schrödinger equation [31]: $u_t = ic_1 u_{xx} + ic_3|u^2|u + ic_5|u|^4 u + [(s_0 + s_2|u|^2)u] = 0$;
(2) Pochhammer–Chree equation [32]: $u_{tt} - u_{ttxx} - (a_1 u + a_3 u^3 + a_5 u^5)_{xx} = 0$;
(3) Burgers-KdV-type equation [33]: $u_t + \alpha u^p u_x + \beta u^{2p} u_x + \gamma u_{xx} + \mu u_{xxx} = 0$;
(4) Generalized Klein–Gordon equation [34]: $u_{tt} - (u_{xx} + u_{yy}) + \alpha^2 u_t + g(uu^*)u = 0$, where g is a polynomial with degree five;
(5) Combined dissipative double-dispersive equation [35]: $u_{tt} - \alpha_1 u_{xx} - \alpha_2 u_{xxt} - \alpha_3 (u)^3_{xx} - \alpha_4 (u)^5_{xx} - \alpha_6 u_{xxxx} + \alpha_7 u_{xxtt} = 0$;
(6) Kundu equation [36]: $iu_t + u_{xx} + \beta|u|^p u + \gamma|u|^{2p} u + i\alpha(|u|^p u)_x + i\rho(|u|^p)_x u = 0$.

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